

# Coordinate noncommutativity in strong non-uniform magnetic fields

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Noncommuting spatial coordinates are studied in connection with the motion of a charged particle in a strong generic magnetic field. We derive a relation involving the commutators of the coordinates, which generalizes the one realized in a strong constant magnetic field. As an application, we discuss the coordinate noncommutativity in a slowly varying magnetic field.

PACS numbers: 11.10.Nx, 11.15.Kc

## I. INTRODUCTION

Noncommutativity of space coordinates has been much studied from various points of view [1, 2, 3]. It arises naturally in string theory, where it is related to the presence of a strong background magnetic-like field. If this is constant, one obtains the more familiar case where the coordinate noncommutativity  $[x^i, x^j]$  is a constant antisymmetric quantity. However, if the background field depends on the spatial coordinates, one would expect the coordinate noncommutativity to be a local function. This possibility has been recently studied in the context of noncommutative gauge field theories [4].

On the other hand, coordinate noncommutativity may also arise in more physical situations involving the motion of electric charges in a strong external magnetic field [5, 6]. When a charged ( $e$ ) and massive ( $m$ ) particle moves on a plane  $(x, y)$  in the presence of a strong constant magnetic field  $B$  pointing along the  $z$ -axis, it has been shown that the noncommutativity of space coordinates is of order of the inverse of the magnetic field:

$$[x, y] = -i\hbar \frac{c}{eB}. \quad (1)$$

An interesting discussion of this behavior, which is related to the fact that the large  $B$  limit corresponds to small  $m$ , has been recently given by Jackiw [7].

Motivated by the above observations, we study in this note the motion of a charged particle in a strong non-uniform magnetic field  $\mathbf{B}(\mathbf{x})$ . Then, we argue that the relation (1) can be generalized to the rotationally symmetric form:

$$[x^i, x^j] = -i\hbar \frac{c}{e} \epsilon^{ijk} \frac{B_k(\mathbf{x})}{B^2(\mathbf{x})}, \quad (i, j, k = 1, 2, 3) \quad (2)$$

which shows that the coordinate noncommutativity is in this case a local function.

This result for the coordinate noncommutativity in non-uniform magnetic fields is derived in section 2. As an application, we study in section 3 the behavior of noncommuting coordinates in a slowly varying magnetic field which is present, for example, in a magnetic mirror.

## II. NONCOMMUTING COORDINATES

In order to derive the relation (2), we consider the equation of motion of a charged particle in a static external magnetic field:

$$m\ddot{\mathbf{x}} = \frac{e}{c} \dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x}) + \mathbf{f}(\mathbf{x}), \quad (3)$$

where  $\mathbf{f}(\mathbf{x})$  represents additional static forces which may be derived from a potential  $V$ :  $\mathbf{f} = -\nabla V$ . In the presence of a strong magnetic field, the Lorentz force term can dominate the kinetic term  $m\ddot{\mathbf{x}}$ , which therefore may be dropped in first approximation. The resulting equation, however, cannot determine all components of the velocity  $\dot{\mathbf{x}}$ , since the projection of  $\dot{\mathbf{x}}$  along  $\mathbf{B}$  is not specified in (3). This is reflected in the equation:

$$(\dot{\mathbf{x}} \times \mathbf{B})_k + \frac{c}{e} f_k = \epsilon_{kij} \dot{x}^i B^j + \frac{c}{e} f_k = 0, \quad (4)$$

in that the antisymmetric matrix  $(\epsilon_k)_{ij}$  does not have an inverse. Multiplying (4) by  $B^k$ , we obtain the consistency condition:

$$B^k f_k = \mathbf{B} \cdot \mathbf{f} = 0. \quad (5)$$

This relation ensures that the net force in the direction of  $\mathbf{B}$  vanishes, which represents a condition necessary to obtain, in the limit  $m \rightarrow 0$ , a consistent set of equations of motion. In fact, since the Lorentz force is orthogonal to the magnetic field, this condition allows us to set the projection of  $m\ddot{\mathbf{x}}$  along  $\mathbf{B}$  equal to zero. The configuration described by equation (5) may be achieved provided the magnetic field is perpendicular to some two-dimensional manifold  $\mathcal{M}$ . Then, if we take the potential  $V$  to be a function defined on  $\mathcal{M}$ ,  $\mathbf{f} = -\nabla V$  will be tangential to this manifold, so that the condition (5) can be satisfied.

One can see in a simple way that the form (2) for the coordinate noncommutativity is consistent with the equation of motion (4). To this end, let us consider the reduced Hamiltonian:

$$H_0 = V(\mathbf{x}) \quad (6)$$

which is obtained in the limit  $m \rightarrow 0$ , by setting the kinematical momentum  $m\dot{\mathbf{x}}$  equal to zero. Then, taking the Poisson bracket of  $x^i$  with  $H_0$  and using the relation:

$$\dot{x}^i = \{x^i, H_0\} = f_j \{x^j, x^i\} \quad (7)$$

one can verify that the equation of motion (4) is satisfied when the brackets which describe noncommuting coordinates are given by the relation (2).

We shall now give a canonical derivation of noncommutativity in the limit  $m \rightarrow 0$ , which is based on the Hamiltonian:

$$H = \frac{\boldsymbol{\pi}^2}{2m} + V(\mathbf{x}) = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + V(\mathbf{x}) \quad (8)$$

where  $\boldsymbol{\pi}$  is the kinematical momentum,  $\mathbf{p}$  is the canonical momentum and  $\mathbf{A}$  denotes the vector potential in the Maxwell theory. In order to be able to set  $m = 0$  in (8), we must impose  $\boldsymbol{\pi} = 0$  as a constraint. This can be implemented using Dirac's method for dealing with constrained systems [8, 9] (for an alternative approach, see reference [10]). Using this method, we consider the constraints:

$$\pi^i = p^i - \frac{e}{c} A^i \approx 0 \quad (i = 1, 2, 3) \quad (9)$$

and evaluate their time evolution using the relation:

$$\dot{\pi}^i = \{\pi^i, H + \lambda_j \pi^j\} = \{\pi^i, V + \lambda_j \pi^j\} = 0 \quad (10)$$

where  $\lambda_j$  represent the Lagrange multipliers in the constrained theory. Using the canonical Poisson brackets, together with the relation:

$$\{\pi^i, \pi^j\} = \frac{e}{c} (\partial^i A^j - \partial^j A^i) = \frac{e}{c} \epsilon^{ijk} B_k, \quad (11)$$

we obtain from (10) the following set of equations involving these multipliers:

$$\epsilon_{ijk} \lambda^j B^j - \frac{c}{e} \frac{\partial V}{\partial x^k} = (\boldsymbol{\lambda} \times \mathbf{B})_k + \frac{c}{e} f_k = 0. \quad (12)$$

This has the same structure as the one of the equation (4), so that we may apply similar considerations as before. Namely, although this system leads to a consistent relation among the Lagrangian multipliers which implies the condition (5), it cannot determine all the  $\lambda^i$  since the projection of  $\boldsymbol{\lambda}$  along  $\mathbf{B}$  is not specified. One can check this in more detail by writing  $\boldsymbol{\lambda}$  in terms of a linear combination, with arbitrary coefficients, of the orthogonal vectors  $\mathbf{B}$ ,  $\mathbf{f}$  and  $\mathbf{B} \times \mathbf{f}$ . Then, from equation (12) it follows that  $\boldsymbol{\lambda}$  must actually have the form:

$$\boldsymbol{\lambda} = \alpha \mathbf{B} - \frac{c}{e} \frac{\mathbf{B} \times \mathbf{f}}{B^2}, \quad (13)$$

so that the coefficient  $\alpha$  remains undetermined. Using this result, the total Hamiltonian in equation (10) can be written in the form:

$$H_t = V + \alpha \mathbf{B} \cdot \boldsymbol{\pi} + \frac{c}{e} \frac{(\mathbf{f} \times \mathbf{B})}{B^2} \cdot \boldsymbol{\pi} \quad (14)$$

We note here that  $c(\mathbf{f} \times \mathbf{B})/eB^2$  represents the drift velocity of the particle due to the force  $\mathbf{f}$ . A well-known example of this velocity is the  $\mathbf{E} \times \mathbf{B}$  drift which is obtained by substituting  $\mathbf{f} = -\nabla V$  in the expression above.

Since the coefficient  $\alpha$  in the Hamiltonian (14) is arbitrary, one may expect that:

$$\phi = \mathbf{B} \cdot \boldsymbol{\pi} \quad (15)$$

would be a first class constraint [8], which commutes with all constraints  $\pi^i$ . This is indeed the case, as one can easily check with the help of equation (11).

Consequently, out of the three constraints  $\pi^i$ , we will be left over with just two second-class constraints, which do not commute. We may take these to be given by the following linear combinations of the  $\pi^i$ :

$$\chi^1 = \mathbf{f} \cdot \boldsymbol{\pi} ; \quad \chi^2 = (\mathbf{B} \times \mathbf{f}) \cdot \boldsymbol{\pi} \quad (16)$$

There is no loss of generality by this choice, since the vectors  $\mathbf{B}$ ,  $\mathbf{f}$  and  $\mathbf{B} \times \mathbf{f}$  are linearly independent.

To proceed with the canonical formalism, we now introduce the Dirac brackets:

$$\{x^i, x^j\}_D = \{x^i, x^j\} - \{x^i, \chi^k\} C_{kl} \{\chi^l, x^j\}, \quad (17)$$

where the matrix  $C_{kl}$  is defined by:

$$C_{kl} \{\chi^l, \chi^i\} = \delta_k^i. \quad (18)$$

From equations (11) and (16) one can check, using the canonical Poisson brackets, that:

$$\{x^i, \chi^1\} = f^i ; \quad \{x^i, \chi^2\} = \epsilon^{ijk} B_j f_k ; \quad \{\chi^1, \chi^2\} = \frac{e}{c} B^2 f^2. \quad (19)$$

Then, with the help of equations (18) and (19), one finds that the Dirac bracket (17) takes the form:

$$\{x^i, x^j\}_D = -\frac{c}{e} \epsilon^{ijk} \frac{B_k(\mathbf{x})}{B^2(\mathbf{x})}. \quad (20)$$

One may pass over to the quantum theory, by taking the commutation relations to correspond to  $i\hbar$  times the Dirac bracket relations. Then, from (20), one can verify the result given in equation (2).

Examples of this type emerge on any 2D (co)adjoint orbit  $\mathcal{M}$  (see [11, 12, 13]), e.g. for a unit sphere  $S^2$  with magnetic monopole in its centre. The monopole magnetic field  $\mathbf{B} = B\mathbf{x}$ , with  $\mathbf{x}^2 = 1$ , gives the Dirac brackets (20) in the form  $\{x^i, x^j\}_D = -\frac{c}{eB} \epsilon^{ijk} x_k$ . For discrete values of  $B = \pm \frac{e}{c} \sqrt{s(s+1)}$ ,  $s$  half-integer, the quantization leads to the well-known fuzzy sphere.

### III. DISCUSSION

The solution (2), which is symmetric under rotations in three dimensions, describes noncommuting spatial coordinates in a generic magnetic field. For consistency, such a noncommutative algebra must satisfy the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \quad (21)$$

In order to show that this identity is satisfied, we note that the Jacobi identity requires the condition:

$$\epsilon_{kij} [x^i, x^j] \partial_l [x^l, x^k] = 0. \quad (22)$$

Then, using the expression (2), we may write this condition in the form:

$$\mathbf{B} \cdot \boldsymbol{\nabla} \times \left( \frac{\mathbf{B}}{B^2} \right) = 0. \quad (23)$$

With the help of the Maxwell equation  $\boldsymbol{\nabla} \times \mathbf{B} = 0$  for the static external magnetic field, we can see that the above equation is indeed satisfied.

As an application of the result (2), let us consider the case of a slowly varying magnetic field in the  $z$ -direction. Such a field occurs in a magnetic mirror [14] which confines the particle's motion in the  $z$ -direction. It may be written in cylindrical coordinates in the form:

$$\mathbf{B} = -\frac{1}{c} \frac{\partial B_z(z)}{\partial z} \hat{\mathbf{e}}_\phi + B_z(z) \hat{\mathbf{e}}_z. \quad (24)$$

where  $\rho B'_z \ll B_z$ . Then, the solution (2) implies the following relations among the noncommuting coordinates:

$$[x, y] = -i\hbar \frac{c}{e} \frac{B_z}{B^2} \quad ; \quad [y, z] = i\hbar \frac{c}{2e} \frac{x B'_z}{B^2} \quad ; \quad [z, x] = i\hbar \frac{c}{2e} \frac{y B'_z}{B^2}. \quad (25)$$

We see that in this case the strongest coordinate noncommutativity occurs in the  $(x, y)$  plane and that the noncommutativity in the  $(x, z)$  and  $(y, z)$  plane is weaker by a factor of order  $\rho B'_z/B_z \ll 1$ .

As is well known [15], in the presence of a constant magnetic field along the  $z$ -direction, the quantum energy levels of a charged particle are given by:

$$E_{n,l} = \frac{eB_z}{2mc} \hbar(2n + |l| - l + 1) + \frac{\hbar^2 k_z^2}{2m}, \quad (26)$$

where  $n = 0, 1, 2, \dots$  and  $\hbar l$  gives the projection of the angular momentum on the  $z$ -axis. The first term in (26) is associated with the motion in the  $(x, y)$  plane, and describes the Landau levels which are infinitely degenerate. The second term gives the translational energy of the particle associated with its motion in the  $z$ -direction.

One can show that the relation (26) may also provide a good approximation for the quantum energy levels of a charged particle in a magnetic mirror, where  $B_z$  is a slowly varying function of  $z$ . In this case, one can see that as the particle drifts along the  $z$ -axis, there will occur a gradual shift of the Landau levels. This shift will be compensated by a corresponding change in the translational energy of the particle, so that its total energy remains conserved. We note that, since the separation between the Landau levels is given by  $\hbar e B_z/mc$ , in a strong magnetic field only the lowest Landau level is relevant. Furthermore, the large  $B_z$  limit is asymptotically equivalent to the limit  $m \rightarrow 0$ . Hence, we may interpret the coordinate noncommutativity (25) as arising in consequence of the fact that our system is constrained to lie in the lowest Landau level.

We wish to thank A. Das and J. C. Taylor for reading the manuscript and the referee for helpful comments. This work was supported by CAPES, CNPq and FAPESP, Brazil.

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